Exercise 53

Find y'' if $x^6 + y^6 = 1$.

Solution

Take the derivative of both sides with respect to x.

$$\frac{d}{dx}(x^6 + y^6) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^6) + \frac{d}{dx}(y^6) = 0$$

$$6x^5 + 6y^5 \cdot \frac{d}{dx}(y) = 0$$

$$6x^5 + 6y^5\frac{dy}{dx} = 0$$

Divide both sides by 6.

$$x^5 + y^5\frac{dy}{dx} = 0 \tag{1}$$

Solve for dy/dx.

$$y^5\frac{dy}{dx} = -x^5$$

$$\frac{dy}{dx} = -\frac{x^5}{y^5}$$

Take the derivative of both sides of equation (1) with respect to x.

$$\frac{d}{dx}\left(x^5 + y^5\frac{dy}{dx}\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^5) + \frac{d}{dx}\left(y^5\frac{dy}{dx}\right) = 0$$

$$5x^4 + \left[\frac{d}{dx}(y^5)\right]\frac{dy}{dx} + y^5\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 0$$

$$5x^4 + \left(5y^4\frac{dy}{dx}\right)\frac{dy}{dx} + y^5\left(\frac{d^2y}{dx^2}\right) = 0$$

$$5x^4 + 5y^4\left(\frac{dy}{dx}\right)^2 + y^5\frac{d^2y}{dx^2} = 0$$

Substitute the formula for dy/dx.

$$5x^4 + 5y^4\left(-\frac{x^5}{y^5}\right)^2 + y^5\frac{d^2y}{dx^2} = 0$$

$$5x^4 + 5y^4\left(\frac{x^{10}}{y^{10}}\right) + y^5\frac{d^2y}{dx^2} = 0$$

$$5x^4 + \frac{5x^{10}}{y^6} + y^5\frac{d^2y}{dx^2} = 0$$

Solve for the term with d^2y/dx^2.

$$y^5\frac{d^2y}{dx^2} = -5x^4 - \frac{5x^{10}}{y^6}$$

$$= \frac{-5x^4y^6 - 5x^{10}}{y^6}$$

$$= \frac{-5x^4(y^6 + x^6)}{y^6}$$

Use the fact that $x^6 + y^6 = 1$.

$$y^5\frac{d^2y}{dx^2} = \frac{-5x^4(1)}{y^6}$$

$$= -\frac{5x^4}{y^6}$$

Therefore, dividing both sides by y^5,

$$\frac{d^2y}{dx^2} = -\frac{5x^4}{y^{11}}.$$